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FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS F/G 12/1  
A SIMPLE MODEL APPLICABLE IN STRUCTURAL RELIABILITY, EXTINCTION--ETC(U)  
NOV 76 E EL-NEWEIHI, F PROSCHAN, J SETHURAMAN AF-AFOSR-2581-74  
FSU-STATISTICS-M392 AFOSR-TR-76-1406 NL

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**A SIMPLE MODEL APPLICABLE IN STRUCTURAL RELIABILITY,  
EXTINCTION OF SPECIES, INVENTORY DEPLETION, AND  
URN SAMPLING, II. SYSTEM LIFELENGTH.**

by <sup>1,2b</sup>  
**E. El-Newehi<sup>a</sup>, F. Proschan<sup>1,2b</sup>, and J. Sethuraman<sup>2b</sup>**  
**University of Kentucky<sup>a</sup> and The Florida State University<sup>b</sup>**

**FSU Statistics Report M392  
AFOSR Technical Report No. 66  
ARO-D Technical Report No. 19**

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**November, 1976  
Department of Statistics  
The Florida State University  
Tallahassee, Florida 32306**

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<sup>1</sup>Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant AFOSR-74-2581C.

<sup>2</sup>Research sponsored by the Army Research Office-Durham under Grant No. DAAG 29-76-0-0238.

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A SIMPLE MODEL APPLICABLE IN STRUCTURAL RELIABILITY,  
EXTINCTION OF SPECIES, INVENTORY DEPLETION, AND  
URAN SAMPLING, II. SYSTEM ELEMENTS.

by E. El-Newehi, F. Proschan, and J. Selhausen,  
University of Kentucky and The Florida State University

AFSC Statistical Report MS92  
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**AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)**

**NOTICE OF TRANSMITTAL TO DDC**

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AFSC 830 A0A



**A Simple Model Applicable in Structural Reliability,  
Extinction of Species, Inventory Depletion, and  
Urn Sampling, II. System Lifelength.**

by

**E. El-Newehi<sup>1</sup>, F. Proschan<sup>2</sup>, and J. Sethuraman<sup>2</sup>**

**ABSTRACT**

This is Part II of a two-part paper devoted to the following model. A series-parallel system consists of  $k$  subsystems in series, subsystem  $i$  containing  $n_i$  components in parallel (denoted  $C_i$  and called cut set  $i$ ),  $i=1, \dots, k$ . After any component in the system fails, the next component failure is equally likely to be any of the components in the system still functioning. In Part I, we computed the probability that a specified cut set fails before any of the others do, and the probability of failure of the cut sets in a specified sequence. We also computed recurrence relations for and various interesting properties of these probabilities.

In the present Part II, we compute the probability distribution, frequency function, and failure rate of the lifelength of series-parallel systems. We also obtain corresponding recurrence relations and finite and asymptotic properties.

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**A Simple Model Applicable in Structural Reliability,  
Extinction of Species, Inventory Depletion, and  
Urn Sampling, II. System Lifelength.**

by

**E. El-Newehi, F. Proschan, and J. Sethuraman**

**1. Introduction and Summary.** This is the second part of a two-part paper devoted to the study of a simple probability model which has applications in reliability theory, inventory theory, the extinction of species, and sampling from urns.

**Reliability Model.** We state the model initially in the reliability context, and then show that the same model can be used in the other areas mentioned. Also, in later sections in presenting the solution and describing its properties, we use the language of reliability, without repeatedly pointing out that the solution and its properties apply in the other areas.

Consider a system consisting of  $k$  subsystems in series, where subsystem  $i$  is a parallel arrangement of  $n_i$  components; such a system is called a series-parallel system. We shall usually refer to the parallel subsystems in a series-parallel system as cut sets. If little is known about the reliability or life distributions of the critical components in the system (as might well be the case in the very early stages of system design), it is not unreasonable to assume all components are equally likely to fail.

In Part I, we raised and answered such questions as:

- (I-1) What is the probability that a given cut set fails before any of the others do?
- (I-2) What is the probability that cut sets fail in a specified sequence?
- (I-3) What properties do these probabilities possess? What inequalities and bounds can be obtained from a knowledge of these properties?



Although the model was originally conceived in a reliability context, it may be applied in a meaningful way in other areas. Consider the following applications of the same model:

**Extinction of Species.** As a concrete example, consider a lake containing in addition to other forms of life,  $k$  species of fish, species 1 containing  $n_1$  specimens,  $i=1, \dots, k$ . The fish are comparable in their vulnerability to capture; fish are caught in succession in a period during which no births occurs (or if new-born fish are caught, they are thrown back into the lake). What is the probability that a given species becomes completely depleted before any of the others do? What is the probability that depletion of species occurs in a specified order?

**Inventory Depletion.**  $k$  types of items are stocked in a depot. Demands for the given types of items are equally likely. The two types of probabilities of depletion in the earlier models and their properties are also of interest in the present model.

**Urn Scheme.** An urn contains  $n_i$  balls of color  $i$ ,  $i=1, \dots, k$ . Balls are drawn in succession, with each of the balls remaining in the urn equally likely to be drawn at each drawing. What is the probability of depletion of a given color first? What is the probability of depletion of colors in a given sequence? What are the properties and bounds for these probabilities?

In the present Part II, we study the lifelength of the series-parallel system - its distribution function, frequency function, and failure rate function.

By system lifelength we mean the number of component failures at the time of the system failure. In addition, we derive interesting and useful properties and bounds for these lifelength functions.

In Section 2, we derive basic formulas for  $P(L(\underline{n}) > \ell)$ , the probability that the system survives the failure of  $\ell$  components, and  $EL(\underline{n})$ , the expected value of system lifelength  $L(\underline{n})$ . Recall that  $\underline{n} = (n_1, \dots, n_k)$ ,  $n_i$  = number of components in cut set  $i$ ,  $i=1, \dots, k$ , and  $k$  = number of cut sets in the series-parallel system. (In Part I, it was more convenient to assume the system consisted of  $k+1$  cut sets labeled  $C_0, C_1, \dots, C_k$ .) We also obtain various useful recurrence relations satisfied by  $P(L(\underline{n}) > \ell)$ ,  $EL(\underline{n})$ , and the probability frequency function  $P(L(\underline{n}) = \ell)$ .

In Section 3, we derive various useful properties of  $P(L(\underline{n}) > \ell)$ ,  $EL(\underline{n})$ , and  $P(L(\underline{n}) = \ell)$ . Thus we show that  $P(L(\underline{n}) > \ell)$  and  $EL(\underline{n})$  are Schur-concave in  $(n_1, \dots, n_k)$ . In addition, we obtain asymptotic properties.

Finally, in Section 4 we obtain properties of the system survival function from the point of view of system aging. Thus we show that the system survival function has the "new better than used" property, i.e.,

$$P(L(\underline{n}) > \ell + m) \leq P(L(\underline{n}) > \ell) P(L(\underline{n}) > m)$$

for all  $\ell=0,1,2,\dots$ ;  $m=0,1,2,\dots$ . We obtain an explicit expression for the system failure rate  $P(L(\underline{n}) = \ell + 1)/P(L(\underline{n}) > \ell)$ . We conjecture it to be an increasing function of  $\ell$ , but are unable to prove the conjecture, except in special cases.

**2. Basic Formulas and Recurrence Relations.** In this section we derive basic expressions for  $P(L(\underline{n}) > \ell)$  and  $EL(\underline{n})$ . We also obtain useful recurrence relations



satisfied by  $P(L(n) > \ell)$ ,  $EL(n)$ , and  $P(L(n) = \ell)$ . With suitable computer programs, these recurrence relations yield numerical values for the quantities of interest. Also the results of this section are used in Section 3 to derive some interesting properties of  $L(n)$ .

Throughout, we shall assume a series-parallel system of  $k$  cut sets  $C_1, \dots, C_k$ , the  $i$ th cut set consisting of  $n_i$  components in parallel,  $i=1, \dots, k$ , unless otherwise stated. We also denote the total number of components  $n_1 + \dots + n_k$  by  $n$ .

Before we state and prove Theorem 2.1, we need the following lemma presented in Part I and proved in Langberg, Proschan, and Quinzi (1977).

**Lemma 2.1.** Let  $(\Omega, \mathcal{A}, P)$  be a probability space and  $A_i \in \mathcal{A}$ ,  $i=1, \dots, m$ , be given events. Then

$$(2.1) \quad P\left(\bigcap_{i=1}^m A_i\right) = \sum_{i=1}^m P(A_i) - \sum_{1 \leq i < j \leq m} P(A_i \cup A_j) + \dots + (-1)^{m-1} P\left(\bigcup_{i=1}^m A_i\right)$$

We may now prove:

**Theorem 2.1.** The survival probability function is given by:

$$(2.2) \quad P(L(n) > \ell) = \begin{cases} \sum_S (-1)^{|S|} \prod_{i \in S} \left[ \sum_{j \in S} \binom{\ell}{j} / \binom{n}{j} \right] & \text{for } \ell \leq n - k \\ 0 & \text{otherwise,} \end{cases}$$

where

$S$  is an arbitrary subset of  $\{1, \dots, k\}$ , possibly the empty subset,

$|S|$  denotes the cardinality of  $S$ ,

$\binom{a}{b} = 0$  for  $b > a$ , and

$\sum_{x \in S} x = 0$  for  $S = \emptyset$ .

**Proof.** Let  $A_i$  denote the event that  $C_i$  has at least one component among the last  $n-l$  failing components,  $i=1, \dots, k$ . Then  $P(L(n) > l) = P(\bigcap_{i=1}^k A_i)$ . By (2.1),

$$\begin{aligned} P(L(n) > l) &= \sum_{i=1}^k P(A_i) - \sum_{1 \leq i < j \leq k} P(A_i \cup A_j) + \dots + (-1)^{k-1} P\left(\bigcup_{i=1}^k A_i\right) \\ &= \sum_{i=1}^k \left[1 - \binom{l}{n_i} / \binom{n}{n_i}\right] - \sum_{1 \leq i < j \leq k} \left[1 - \binom{l}{n_i + n_j} / \binom{n}{n_i + n_j}\right] + \dots \\ &\quad + (-1)^{k-1} \left[1 - \binom{l}{n} / \binom{n}{n}\right] \\ &= \binom{k}{1} - \binom{k}{2} + \dots + (-1)^{k-1} - \sum_{i=1}^k \binom{l}{n_i} / \binom{n}{n_i} + \sum_{1 \leq i < j \leq k} \binom{l}{n_i + n_j} / \binom{n}{n_i + n_j} + \dots \\ &\quad + (-1)^k \binom{l}{n} / \binom{n}{n}. \end{aligned}$$

Since  $\binom{k}{1} - \binom{k}{2} + \dots + (-1)^{k-1} = 1$ , it follows that

$$P(L(n) > l) = \sum_{S \subseteq \{1, \dots, k\}} (-1)^{|S|} \left[ \binom{l}{\sum_{i \in S} n_i} / \binom{n}{\sum_{i \in S} n_i} \right]. \quad ||$$

**Corollary 2.1.** The expected system lifelength is given by:

$$(2.3) \quad EL(n) = \sum_{l=0}^{n-k} \left\{ \sum_{S \subseteq \{1, \dots, k\}} (-1)^{|S|} \left[ \binom{l}{\sum_{i \in S} n_i} / \binom{n}{\sum_{i \in S} n_i} \right] \right\}$$

**Proof.** The result follows immediately from (2.2) and the fact that

$$EL(n) = \sum_{l=0}^{n-k} P(L(n) > l). \quad ||$$



**Remark.** For the trivial case  $k=1$ , (2.2) and (2.3) yield the obvious results that  $L(n_1) = n_1$  with probability one and  $EL(n_1) = n_1$ .

An expression with alternating signs is inconvenient for certain purposes.

Thus we next represent  $P(L(n) > \ell)$  and  $EL(n)$  as sums of positive terms only.

**Theorem 2.2.** An alternative formula for  $P(L(n) > \ell)$  is given by:

$$(2.4) \quad P(L(n) > \ell) = \sum_{\substack{x_1 \geq 1 \\ \sum x_i = n - \ell}} \prod_{i=1}^k \frac{\binom{n_i}{x_i}}{\binom{n}{n-\ell}} \quad \text{for } \ell \leq n - k,$$

and

$$(2.5) \quad EL(n) = \sum_{\sum x_i \leq n} \prod_{i=1}^k \frac{\binom{n_i}{x_i}}{\binom{n}{x_i}}.$$

**Proof.** The system continues to function after  $\ell$  of its components have failed if and only if each of the cut sets  $C_1, \dots, C_k$  has at least one of its components still functioning. This implies (2.4).

Since  $EL(n) = \sum_{\ell} P(L(n) > \ell)$ , (2.5) follows from (2.4). ||

**Remark.** The relation (2.4) may also be seen readily by considering the following equivalent combinatorial problem. Assume that urn  $i$  has  $n_i$  balls,  $i=1, \dots, k$ . The balls are drawn at random one at a time, the  $n!$  possible orders of the drawn balls being equally likely. Then the probability that none of the urns is empty after the first  $\ell$  balls have been drawn is  $P(L(n) > \ell)$ , given by (2.4).

The following theorem gives alternative expressions for  $P(L(n) > \ell)$  and  $EL(n)$ . These various expressions will be used to prove interesting properties of  $L(n)$ .

**Theorem 2.3.**

$$(2.6) \quad P(L(n) > \ell) = \sum_{\substack{x_1 \geq 1 \\ \sum x_i = n - \ell}} \frac{\ell!}{(n_1 - x_1)! \cdots (n_k - x_k)!} \frac{(n - \ell)!}{x_1! \cdots x_k!} / \frac{n!}{n_1! \cdots n_k!}$$

for  $\ell \leq n - k$ , and

$$(2.7) \quad EL(n) = \sum_{\substack{x_1 \geq 1 \\ \sum x_i \leq n}} \frac{(n - \sum x_i)!}{(n_1 - x_1)! \cdots (n_k - x_k)!} \frac{\sum x_i}{x_1! \cdots x_k!} / \frac{n!}{n_1! \cdots n_k!}.$$

**Proof.** The numerator of (2.6) represents the number of ways at least one component from each cut set survives past the  $\ell^{\text{th}}$  component failure; the denominator represents the number of arrangements of the  $n$  component failures without distinguishing among components from the same cut set. ||

For the special case of  $k=2$ , the expressions given for  $P(L(n) > \ell)$ ,  $P(L(n) > \ell)$ , and  $EL(n)$  reduce readily to simple closed forms. Not only are these results useful for the special case of  $k=2$ , but they will also be directly utilized to derive interesting properties for the general case in the next section.

**Theorem 2.4.** Consider a series-parallel system consisting of two cut sets  $C_1$  and  $C_2$  having  $n_1$  and  $n_2$  components respectively; let  $n = n_1 + n_2$ . Then

$$(2.8) \quad P(L(n_1, n_2) > \ell) = 1 - \left[ \binom{\ell}{n_1} + \binom{\ell}{n_2} \right] / \binom{n}{n_1}, \quad \ell \leq n - 2,$$

and

$$(2.9) \quad EL(n_1, n_2) = \frac{n_1}{n_1 + 1} \cdot \frac{n_2}{n_2 + 1} \cdot (n + 2).$$



**Proof.** Note that  $\binom{\ell}{n_1} / \binom{n}{n_1}$  is the probability that  $C_1$  fails during the first  $\ell$  failures,  $i=1,2$ . Since both  $C_1$  and  $C_2$  cannot both fail during the first  $\ell (\leq n-2)$  failures, then (2.8) follows.

To prove (2.9), write

$$\begin{aligned} EL(n_1, n_2) &= \sum_{\ell=0}^{n-1} P(L(n_1, n_2) > \ell) = \sum_{\ell=0}^{n-1} \left[ 1 - \left\{ \binom{\ell}{n_1} + \binom{\ell}{n_2} \right\} / \binom{n}{n_1} \right] \\ &= n - \left\{ \binom{n}{n_1+1} / \binom{n}{n_1} \right\} - \left\{ \binom{n}{n_2+1} / \binom{n}{n_2} \right\} = n - \frac{n_2}{n_1+1} - \frac{n_1}{n_2+1} \\ &= \frac{n_1}{n_1+1} \cdot \frac{n_2}{n_2+1} \cdot (n+2). \quad || \end{aligned}$$

**Theorem 2.5.** For a series-parallel system consisting of two cut sets  $C_1$  and  $C_2$  with  $n_1$  and  $n_2$  components respectively, we have:

$$(2.10) \quad P(L(n_1, n_2) = \ell) = \left[ \binom{\ell-1}{n_1-1} / \binom{n}{n_1} \right] + \left[ \binom{\ell-1}{n_2-1} / \binom{n}{n_2} \right]$$

for  $\min(n_1, n_2) \leq \ell \leq n-1$ .

**Proof.** Let  $A_i$  denote the event that cut set  $C_i$  fails at time  $\ell$ ,  $i=1,2$ . Then

$$P(L(n_1, n_2) = \ell) = P(A_1) + P(A_2) = \left[ \binom{\ell-1}{n_1-1} / \binom{n}{n_1} \right] + \left[ \binom{\ell-1}{n_2-1} / \binom{n}{n_2} \right]. \quad ||$$

Next we present recurrence relations satisfied by  $P(L(\underline{n}) > \ell)$ ,  $P(L(\underline{n}) = \ell)$ , and  $EL(\underline{n})$ . Let  $(\underline{n})_1$  denote the  $(k-1)$ -tuple obtained from  $\underline{n}$  by deleting  $n_1$ .

**Theorem 2.6.**

$$(a) \quad P(L(\underline{n}) > \ell) = \sum_{i=1}^k \frac{n_i}{n} \left[ \sum_{\substack{x_j \geq 1, j \neq i \\ \sum x_j = n-1-\ell}} \binom{n_1}{x_1} \dots \binom{n_{i-1}}{x_{i-1}} \dots \binom{n_k}{x_k} / \binom{n-1}{\ell} \right],$$

$$(b) \quad P(L(\underline{n}) = \ell) = \sum_{i=1}^k \left[ \binom{\ell-1}{n_i-1} / \binom{n}{n_i} \right] P[L(\underline{n})_i > \ell - n_i]$$

$$(c) \quad EL(\underline{n}) = \sum_{i=1}^k \frac{n_i}{n} EL(n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_k) + 1$$

**Proof.** (a) Let  $A_i$  denote the event that the last component to fail is from  $C_i$ ,  $i=1, \dots, k$ . The result in (a) follows immediately from the law of total probability, namely:

$$P(L(\underline{n}) > \ell) = \sum_{i=1}^k P(L(\underline{n}) > \ell | A_i) P(A_i).$$

(b) Let  $B_i$  denote the event that cut set  $C_i$  fails at time  $\ell$ ,  $i=1, \dots, k$ .

In a similar fashion, we have:

$$P(L(\underline{n}) = \ell) = \sum_{i=1}^k P(L(\underline{n}) = \ell | B_i) P(B_i),$$

and the desired result follows.

(c) Since  $L(\underline{n}) = L(n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_k) + 1$  with probability  $n_i/n$ ,  $i=1, \dots, k$ , then

$$\begin{aligned} EL(\underline{n}) &= \sum_{i=1}^k \frac{n_i}{n} E[L(n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_k) + 1] \\ &= \sum_{i=1}^k EL(n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_k) + 1. \end{aligned}$$



**Remark.** The recurrence relations (a) and (c) in Theorem 2.6 provide expressions for  $P(L(n) > \ell)$  and  $EL(n)$  in terms of similar quantities obtained by reducing the number of components in one of the  $k$  cut sets by one. Relation (b) provides an expression for  $P(L(n) = \ell)$  in terms of  $P(L(n)_1 > \ell - n_1)$  for systems with fewer cut sets.

**3. Properties of the Lifelength  $L(n)$ .** Using the results of Section 2, we now obtain various properties of  $P(L(n) > \ell)$  and  $EL(n)$ . These results express in precise form several intuitively obvious properties of  $P(L(n) > \ell)$  and  $EL(n)$ . These properties may be helpful in designing series-parallel systems so as to maximize both  $P(L(n) > \ell)$  and  $EL(n)$ . We start with the obvious property:

**Theorem 3.1.** Both  $P(L(n) > \ell)$  and  $EL(n)$  are symmetric functions of  $n_1, \dots, n_k$ .

As one might expect, the more components there are in the cut sets, the more likely it is that the system survives the failure of  $\ell$  components. This is made precise in:

**Theorem 3.2.** For all  $\ell \geq 0$ ,  $P(L(n) > \ell)$  is monotonically increasing in  $n_i$ ,  $i=1, \dots, k$ .

**Proof.** It suffices to show that  $P(L(n_1+1, n_2, \dots, n_k) > \ell) \geq P(L(n_1, \dots, n_k) > \ell)$ .

Let  $L_1, \dots, L_n$  be the exchangeable random variables representing component lifelengths (i.e.,  $L_i$  represents the number of component failures preceding the failure of the  $i^{\text{th}}$  component). Let  $L_{n+1}$  denote the lifelength of the additional component in  $C_1$ . Let  $\tilde{L}_j = \max_{1 \leq i \in C_j} \{L_i\}$ ,  $j=1, \dots, k$ . Then

$L(n) = \sum_{i=1}^n I_{[\min(\tilde{L}_1, \dots, \tilde{L}_k) \geq L_i]}$ . Also, let  $\tilde{L}'_1 = \max(\tilde{L}_1, L_{n+1})$ ; then

$L(n_1 + 1, \dots, n_k) = \sum_{i=1}^{n+1} I_{[\min(\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_k) \geq L_1]}$ . Obviously  $L(n_1 + 1, \dots, n_k) \geq L(n_1, \dots, n_k)$  for every sample outcome. The desired result follows immediately. ||

**Corollary 3.1.**  $EL(\underline{n})$  is monotonically increasing in  $n_i, i=1, \dots, k$ .

**Proof.** Theorem 3.2 and the fact that  $X \leq Y$  a.s. imply that  $EX \leq EY$ . ||

**Remark.** For the special case of  $k=2$ , a direct proof for Theorem 3.2 and Corollary 3.1 using (2.8) and (2.9) is easy.

The concepts of majorization and Schur functions have been applied to develop a variety of useful inequalities in many branches of mathematics and statistics. Majorization (see Definition 3.1 below) is a partial ordering in  $R_k$ , the  $k$ -dimensional Euclidean space. A Schur function is a function that is monotone with respect to this partial ordering. Many well known inequalities arising in probability and statistics are equivalent to the statement that certain functions are Schur functions. Theorem 3.3 and Corollary 3.2 below show that for a fixed  $\ell$ ,  $P(L(\underline{n}) > \ell)$  and  $EL(\underline{n})$  are Schur functions in  $n_1, \dots, n_k$ . Before presenting these results, we give definitions of majorization and Schur functions. We use the notation that for a given vector  $\underline{x} = (x_1, \dots, x_k)$ , the decreasing rearrangement of the coordinates is denoted by  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[k]}$ .

**Definition 3.1.** A vector  $\underline{x}$  is said to majorize a vector  $\underline{x}'$  (in symbols  $\underline{x} \stackrel{M}{\geq} \underline{x}'$ ) if

$$\sum_{i=1}^j x_{[i]} \geq \sum_{i=1}^j x'_{[i]}, \quad j = 1, \dots, k-1,$$

and

$$\sum_{i=1}^k x_{[i]} = \sum_{i=1}^k x'_{[i]}.$$



Note that if  $\underline{x}'$  is a permutation of  $\underline{x}$ , then  $\underline{x} \succeq \underline{x}'$  and  $\underline{x}' \succeq \underline{x}$ .

A useful characterization of majorization is given by Hardy, Littlewood, and Pólya (1952), p. 47.

**Lemma 3.1.**  $\underline{x} \succeq \underline{x}'$  if and only if there exists a finite number, say  $r$ , of vectors  $\underline{x}^{(1)}, \dots, \underline{x}^{(r)}$  such that  $\underline{x} = \underline{x}^{(1)} \succeq \underline{x}^{(2)} \succeq \dots \succeq \underline{x}^{(r)} = \underline{x}'$  and such that  $\underline{x}^{(i)}$  and  $\underline{x}^{(i+1)}$  differ in two coordinates only,  $i=1, 2, \dots, r-1$ .

**Definition 3.2.** A function  $f: R_k \rightarrow R$  is said to be Schur-convex (Schur-concave) if  $f(\underline{x}) \geq (\leq) f(\underline{x}')$  whenever  $\underline{x} \succeq \underline{x}'$ . Functions which are either Schur-convex or Schur-concave are called Schur-functions. Note that a Schur function  $f$  is necessarily permutation-invariant (symmetric); that is,  $f(\underline{x}) = f(\underline{x}')$  whenever  $\underline{x}'$  is a permutation of  $\underline{x}$ .

The following are examples of Schur functions:

$f_1(\underline{x}) = \prod_{i=1}^k \phi(x_i)$  is Schur-convex (Schur-concave) if  $\phi$  is log-convex (log-concave).

$f_2(\underline{x}) = \prod_{i=1}^j x_{[i]}$  is Schur-convex,  $1 \leq j \leq k$ .

$f_3(\underline{x}) = \prod_{i=1}^k x_i$  is both Schur-convex and Schur-concave.

We may now state and prove:

**Theorem 3.3.** For all  $\ell$ ,  $P(L(\underline{n}) > \ell)$  is Schur-concave in  $(n_1, \dots, n_k)$ .

**Proof.** By Lemma 3.1 it suffices to show that:

$$P(L(n_1-1, n_2+1, n_3, \dots, n_k) > \ell) \geq P(L(\underline{n}) > \ell),$$

where  $n_1 > n_2$ .

First we prove the theorem for  $k = 2$ ; i.e., we show that  $P(L(n_1-1, n_2+1) > m) \geq P(L(n_1, n_2) > m)$ . By (2.4) we have  $P(L(n_1, n_2) > \ell) = 1 - \left[ \frac{\binom{n_1}{n-\ell} + \binom{n_2}{n-\ell}}{\binom{n}{n-\ell}} \right]$ ,

where  $n = n_1 + n_2$ . Let  $n - m = a$ ; then for  $n_1 > n_2$  we have:

$$\binom{n_1}{n} + \binom{n_2}{a} = \binom{n_1-1}{a} + \binom{n_1-1}{a-1} + \binom{n_2}{a} \geq \binom{n_1-1}{a} + \binom{n_2}{a-1} + \binom{n_2}{a} = \binom{n_1-1}{a} + \binom{n_2+1}{a}.$$

Therefore  $P(L(n_1, n_2) > m) \leq P(L(n_1-1, n_2+1) > m)$ , which proves the result for  $k = 2$ .

For the general case, we have by (2.4):

$$P(L(\underline{n}) > \ell) = \sum_{\substack{x_1 \geq 1 \\ \sum x_j = n - \ell}} \frac{\binom{n_1}{x_1} \cdots \binom{n_k}{x_k}}{\binom{n}{n-\ell}} \\ = \sum_{\ell'} \left\{ \sum_{\substack{x_j \geq 1, j \geq 3 \\ \sum_{j \geq 3} x_j = n - \ell + \ell'}} \frac{\left[ \binom{n_3}{x_3} \cdots \binom{n_k}{x_k} \binom{n_1+n_2}{\ell'} \right]}{\binom{n}{n-\ell}} P[L(n_1, n_2) > \ell'] \right\}.$$

We have proved  $P(L(n_1, n_2) > \ell')$  is Schur-concave for all  $\ell'$ . Since a nonnegative linear combination of Schur-concave functions is a Schur-concave function, then  $P(L(\underline{n}) > \ell)$  is a Schur-concave function in  $(n_1, \dots, n_k)$ . ||

**Remark.** An alternative proof of Theorem 3.3 can be constructed using a preservation theorem of Proschan and Sethuraman (1977) (Theorem 1.1).

**Corollary 3.2.**  $EL(\underline{n})$  is Schur-concave in  $(n_1, \dots, n_k)$ .



**Proof.** As in Theorem 3.3, it suffices to show that if  $n_1 > n_2$ , then  $EL(n_1-1, n_2+1, n_3, \dots, n_k) \geq EL(\underline{n})$ . By Theorem 3.3,  $L(n_1-1, n_2+1, \dots, n_3, \dots, n_k)$  is stochastically larger than  $L(\underline{n})$ , which implies that  $EL(n_1-1, n_2+1, \dots, n_k) \geq EL(n_1, n_2, \dots, n_k)$ . ||

**Remark 1.** The results in Theorem 3.3 and Corollary 3.2 are intuitively quite reasonable. Roughly speaking, the results state that for a fixed total number  $\sum n_i$  of components, the more homogeneous are the cut set sizes, the larger (stochastically) is  $L(\underline{n})$ .

**Remark 2.** Assume  $n_1 + \dots + n_k = kr$ , then obviously  $(n-k+1, \underbrace{1, \dots, 1}_{(k-1) \text{ times}}) \stackrel{n}{\geq} (n_1, \dots, n_k) \stackrel{n}{\geq} (\underbrace{r, \dots, r}_{k \text{ times}})$ . Since  $P(L(\underline{n}) > \ell)$  and  $EL(\underline{n})$  are Schur-concave functions, we immediately get upper and lower bounds for them.

Next we obtain some asymptotic properties of  $P(L(\underline{n}) > \ell)$  and  $EL(\underline{n})$ .

**Theorem 3.4.** For all  $1 \leq i \leq k$  and for all  $\ell = 0, 1, 2, \dots$ , we have:

$$P(L(\underline{n}) > \ell) \rightarrow 1 \text{ as } n_i \rightarrow \infty.$$

**Proof.** Let  $m = n_1 + \dots + n_{i-1} + n_{i+1} + \dots + n_k$ . For  $n_i$  large enough we have:

$$1 \geq P(L(\underline{n}) > \ell) \geq \frac{\binom{n_i}{\ell}}{\binom{n_i+m}{\ell}} \rightarrow 1 \text{ as } n_i \rightarrow \infty. \quad ||$$

**Remark.** It follows immediately from the above theorem and the monotonicity of  $P(L(\underline{n}) > \ell)$  that  $\lim_{\substack{n_i \rightarrow \infty \\ i \in S}} P(L(\underline{n}) > \ell) = 1$ , where  $S$  is a non-empty subset of  $\{1, \dots, k\}$ .

**Theorem 3.5.** Let  $S$  be a non-empty subset of  $\{1, \dots, k\}$ . Then  $\lim_{n_1 \rightarrow \infty} EL(n) = \infty$ .  
 $1 \in S$

**Proof.** It suffices to show that  $\lim_{n_1 \rightarrow \infty} EL(n) = \infty$ . Write

$$EL(n) = \sum_{\ell=0}^{n-k} P(L(n) > \ell) = \sum_{\ell=0}^{\infty} P(L(n) > \ell) I_{\{0, \dots, 0, n-k\}}(\cdot),$$

where  $I_A(\cdot)$  denotes the indicator function of the set  $A$ . As  $n_1 \rightarrow \infty$ , we have

$P(L(n) > \ell) I_{\{0, \dots, 0, n-k\}}(\cdot) \rightarrow 1$ , and by the monotone convergence theorem,

$$EL(n) \rightarrow \infty. \quad ||$$

**4. Properties Based on Notions of Aging.** In this section we explore notions of system aging analogous to those discussed for general life distributions in Barlow and Proschan (1975), Chaps. 3 and 6. Roughly speaking, we compare the lifelength of a "fresh" series-parallel system with its remaining lifelength after it has survived the failure of  $\ell$ , say, of its components.

Throughout this section, let  $\bar{F}(\ell) = 1 - F(\ell)$  denote the survival probability  $P(L(n) > \ell)$  of the system. Also let  $A_r$  denote the set  $\{(x_1, \dots, x_k) : x_i \geq 1, i = 1, \dots, k, \text{ and } \sum_{i=1}^k x_i = r\}$ .

**Theorem 4.1.** For all nonnegative integers  $\ell$  and  $m$ , we have:

$$(4.1) \quad \bar{F}(\ell + m) \leq \bar{F}(\ell) \bar{F}(m).$$

**Proof.** Let  $\underline{X}_\ell = (X_{\ell_1}, \dots, X_{\ell_k})$ , where  $X_i$  is the random number of components remaining in cut set  $C_i$ ,  $i=1, \dots, k$ , after  $\ell$  component failures. When  $L > \ell$ , notice that  $\underline{X}_\ell \in A_{n-\ell}$ ,  $\ell = 1, 2, \dots$ .



$$\begin{aligned}\bar{F}(\ell+m)/\bar{F}(\ell) &= P(L(\underline{n}) > \ell + m | L(\underline{n}) > \ell) \\ &= \sum_{\underline{x} \in A_{n-\ell}} P(L(\underline{n}) > \ell + m | \underline{X}_\ell = \underline{x}, L(\underline{n}) > \ell) P(\underline{X}_\ell = \underline{x} | L(\underline{n}) > \ell)\end{aligned}$$

Obviously,  $P(L(\underline{n}) > \ell + m | \underline{X}_\ell = \underline{x}, L(\underline{n}) > \ell) = P(L(\underline{x}) > m)$ . By Theorem 3.2 we

have  $P(L(\underline{x}) > m) \leq P(L(\underline{n}) > m)$  since  $x_i \leq n_i, i=1, \dots, k$ . Thus

$$\bar{F}(\ell + m)/\bar{F}(\ell) \leq P(L(\underline{n}) > m) \sum_{\underline{x} \in A_{n-\ell}} P(\underline{X}_\ell = \underline{x} | L(\underline{n}) > \ell) = P(L(\underline{n}) > m) = \bar{F}(m). \text{ Thus}$$

$$\bar{F}(\ell + m) \leq \bar{F}(\ell) \bar{F}(m). \quad ||$$

**Remark.** A life distribution  $F$  for which (4.1) holds is said to be new better than used (NBU). See Barlow and Proschan (1975), Chap. 6, for a discussion of NBU life distributions and their properties.

Recall that  $f(\ell) = P(L(\underline{n}) = \ell)$  is the probability that the system fails with the failure of the  $\ell^{\text{th}}$  failing component. We define the conditional failure rate  $r(\ell)$  of the system by  $r(\ell) = f(\ell)/\bar{F}(\ell-1)$ , so that  $r(\ell)$  denotes the conditional probability that the system fails upon the failure of the  $\ell^{\text{th}}$  component given that it has survived the failure of the first  $\ell-1$  components. When  $r(\ell)$  is a monotonically increasing function in  $\ell$ , the distribution function  $F$  is said to be an increasing failure rate (IFR) distribution.

**Conjecture.** For a series-parallel system, the life distribution  $F$  is IFR.

Next we prove the conjecture for two special cases. First we prove it for the case  $k=2$ :

**Theorem 4.2.** Let  $C_1$  and  $C_2$  be two cut sets forming a series-parallel system having  $n_1$  and  $n_2$  components respectively. Then  $r(\ell)$  is an increasing function of  $\ell$ .

**Proof.** By Theorem 2.5 we have:

$$f(\ell) = \left\{ \binom{\ell-1}{n_1-1} + \binom{\ell-1}{n_2-1} \right\} / \binom{n_1+n_2}{n_1} \text{ for } \min(n_1, n_2) \leq \ell \leq n-1.$$

Obviously,  $f(\ell)$  is increasing, while  $\bar{F}(\ell-1)$  is decreasing. Thus the ratio  $r(\ell)$  is increasing. ||

We extend the above result in:

**Theorem 4.3.** For the series-parallel system let  $n_3 = n_4 = \dots = n_k = 1$ . Then  $F$  is IFR.

**Proof.** We use the fact that  $r(\ell)$  is an increasing function if and only if  $\bar{F}(\ell)/\bar{F}(\ell-1)$  decreases. By (2.4):

$$\begin{aligned} \bar{F}(\ell) &= P(L(n_1, n_2, 1, \dots, 1) > \ell) = \sum_{x_1 \geq 1, x_2 \geq 1} \binom{n_1}{x_1} \binom{n_2}{x_2} / \binom{n_1+n_2+k-2}{\ell} \\ &= H(\ell) \bar{G}(\ell), \end{aligned}$$

where  $H(\ell) = \binom{n_1+n_2}{\ell} / \binom{n_1+n_2+k-2}{\ell}$  and  $\bar{G}(\ell) = P(L(n_1, n_2) > \ell)$ . By Theorem 4.2,  $\bar{G}(\ell)/\bar{G}(\ell-1)$  is decreasing. Also it is easy to verify that  $H(\ell)/H(\ell-1)$  is decreasing. It follows that  $\bar{F}(\ell)/\bar{F}(\ell-1)$  is decreasing, implying that  $r(\ell)$  is increasing. ||

Next we express  $r(\ell)$  in the general case:

**Theorem 4.4.**

$$r(\ell+1) = \frac{\sum_{x_1 \geq 1} \binom{n_1}{x_1} \prod_{i=1}^k \binom{n_i}{x_i} / \binom{n_1+n_2+\dots+n_k}{\ell+1}}{\sum_{x_1 \geq 1} \binom{n_1}{x_1} \prod_{i=1}^k \binom{n_i}{x_i} / \binom{n_1+n_2+\dots+n_k}{\ell}}.$$



**Proof.** Let  $\underline{X} = (X_1, \dots, X_k)$ , where  $X_i$  is the random number of components remaining in  $C_i$ ,  $i=1, \dots, k$ . Then

$$\begin{aligned}
 r(\ell + 1) &= P(L(\underline{n}) = \ell + 1 | L(\underline{n}) > \ell) \\
 &= \sum_{\underline{x} \in A_{n-\ell}} P(L(\underline{n}) = \ell + 1 | \underline{X} = \underline{x}) P(\underline{X} = \underline{x} | L(\underline{n}) > \ell) \\
 &= \sum_{\underline{x} \in A_{n-\ell}} \left( \frac{1}{n-\ell} \prod_{i=1}^k I_{[x_i=1]} \right) \prod_{i=1}^k \binom{n_i}{x_i} / \sum_{\underline{x} \in A_{n-\ell}} \prod_{i=1}^k \binom{n_i}{x_i}. \quad ||
 \end{aligned}$$

**5. Additional Problems.** It seems highly reasonable to believe that the failure rate for the distribution of the lifelength of a series-parallel system is increasing. The intuitive basis for this belief is that as the system continues to survive additional component failures, the number of cut sets of size one increases while the number of functioning components remaining decreases. Since the failure rate is precisely the number of cut sets of size one divided by the number of functioning components, it seems very likely that the failure rate is an increasing function. We are attempting to prove this important conjecture.

We are also studying the life lengths corresponding to more general classes of systems in which cut sets overlap; i.e., the same component may appear in more than one cut set. The most general class of systems of this type is the class of coherent systems. (See Barlow and Proschan, 1975, Chapter 1.) For this general class, the corresponding results concerning lifelength would be much more complicated. We are currently considering some special cases of the series-parallel system in which:

- (a) a single component appears in more than one cut set,
- (b) several components appear in each of two cut sets.

Other directions for generalization include:

- (c) the lifelength of systems other than series-parallel systems,
- (d) components within a cut set have the same reliability; component reliability varies among cut sets.

For each of these models, it would be desirable to obtain exact expressions, bounds, and qualitative properties for system lifelength.

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19

**SECURITY CLASSIFICATION OF THIS PAGE**

**REPORT DOCUMENTATION PAGE**

<b>1. REPORT NUMBER</b> FSU No. M392 AFOSR No. 66 ARO-D No. 19	<b>2. GOVT ACCESSION NO.</b>	<b>3. RECIPIENT'S CATALOG NUMBER</b>
<b>4. TITLE</b> A Simple Model Applicable in Structural Reliability, Extinction of Species, Inventory Depletion, and Urn Sampling, II. System Lifelength.		<b>5. TYPE OF REPORT &amp; PERIOD COVERED</b> Technical Report
<b>7. AUTHOR(s)</b> E. El-Newehi F. Proschan J. Sethuraman		<b>6. PERFORMING ORG. REPORT NUMBER</b> FSU Statistics Report M392
<b>9. PERFORMING ORGANIZATION NAME &amp; ADDRESS</b> The Florida State University Department of Statistics Tallahassee, Florida 32306		<b>8. CONTRACT OR GRANT NUMBER(s)</b> AFOSR-74-2581C DAAG29-76-G-0238
<b>11. CONTROLLING OFFICE NAME &amp; ADDRESS</b> The U.S. Air Force Air Force Office of Scientific Research Bolling Air Force Base, D. C. 20332  U.S. Army Research Office-Durham P. O. Box 12211 Research Triangle Park, N.C. 27709		<b>10. PROGRAM ELEMENT, PROJECT, TASK AREA &amp; WORK UNIT NUMBERS</b>
<b>14. MONITORING AGENCY NAME &amp; ADDRESS (if different from Controlling Office)</b>		<b>12. REPORT DATE</b> November 1976
		<b>13. NUMBER OF PAGES</b> 19
		<b>15. SECURITY CLASS (of this report)</b> Unclassified
		<b>15a. DECLASSIFICATION/DOWNGRADING SCHEDULE</b>
<b>16. DISTRIBUTION STATEMENT (of this report)</b> Approved for public release; distribution unlimited.		
<b>17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)</b>		
<b>18. SUPPLEMENTARY NOTES</b>		

UNCLASSIFIED

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SECURITY CLASSIFICATION OF THIS PAGE

19. KEY WORDS

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Structural reliability, system lifelength, series-parallel system, cut set, extinction of species, inventory depletion, urn sampling, Schur-concavity.

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20. ABSTRACT

This is Part II of a two-part paper devoted to the following model. A series-parallel system consists of  $k$  subsystems in series, subsystem  $i$  containing  $n_i$  components in parallel (denoted  $C_i$  and called cut set  $i$ ),  $i=1, \dots, k$ . After any component in the system fails, the next component failure is equally likely to be any of the components in the system still functioning. In Part I, we computed the probability that a specified cut set fails before any of the others do, and the probability of failure of the cut sets in a specified sequence. We also computed recurrence relations for and various interesting properties of these probabilities.

In the present Part II, we compute the probability distribution, frequency function, and failure rate of the lifelength of series-parallel systems. We also obtain corresponding recurrence relations and finite and asymptotic properties.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER <b>AFOSR-TR-76-1406</b>	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) <b>A SIMPLE MODEL APPLICABLE IN STRUCTURAL RELIABILITY, EXTINCTION OF SPECIES, INVENTORY DEPLETION, AND URN SAMPLING. II. SYSTEM LIFELENGTH.</b>	5. TYPE OF REPORT & PERIOD COVERED <b>Interim rept.</b>		
6. AUTHOR(s) <b>E. El-Newehi, F. Proschan J. Sethuraman</b>	7. CONTRACT OR GRANT NUMBER(s)		
8. PERFORMING ORGANIZATION NAME AND ADDRESS <b>Florida State University Department of Statistics Tallahassee, Florida 32306</b>	9. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBER <b>61102F 2304 A5</b>		
10. CONTROLLING OFFICE NAME AND ADDRESS <b>Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332</b>	11. REPORT DATE <b>Nov 76</b>		
12. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	13. NUMBER OF PAGES <b>19</b>		
	14. SECURITY CLASS. (of this report) <b>UNCLASSIFIED</b>		
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE		
16. DISTRIBUTION STATEMENT (of this Report) <b>Approved for public release; distribution unlimited.</b>			
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)			
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) <b>structural reliability, system lifelength, series-parallel system, cut set, extinction of species, inventory depletion, urn sampling, Schur concavity</b>			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) <b>This is part II of a two-part paper devoted to the following model. series-parallel system consists of k subsystems in series, subsystem i containing <math>n_i</math> components in parallel (denoted <math>C_i</math> and called cut set i), i=1,...,k. After any component in the system fails, the next component failure is equally likely to be any of the components in the system still</b>			

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SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

## 20 Abstract

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→ In the present Part II, we compute the probability distribution, frequency function, and failure rate of the lifelength of series-parallel systems. We also obtain corresponding recurrence relations and finite and asymptotic properties.

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